# 3 More geometry

## 3.1 Sets on the plane

So I explained in the last lecture that it is quite important to see the geometry of the usual real plane when we talk about complex numbers. From now on I will be assuming this fact without explicitly mentioning it. In particular, I will need to be comfortable with various sets of complex numbers, which are of course are just subsets of the real plane. Here are a few examples.

**Example 3.1.** What set on the plane $\mathbf{R}^2$ is defined by the condition

$$\operatorname{Im} z^2 > 2?$$

I have $z^2 = (x + \mathrm{i}y)^2 = (x^2 + y^2) + \mathrm{i}2xy$, and therefore $\operatorname{Im} z^2 = 2xy > 2$ or, finally, $xy > 1$, which is the set of points above and under the hyperbola $xy = 1$, see the figure.

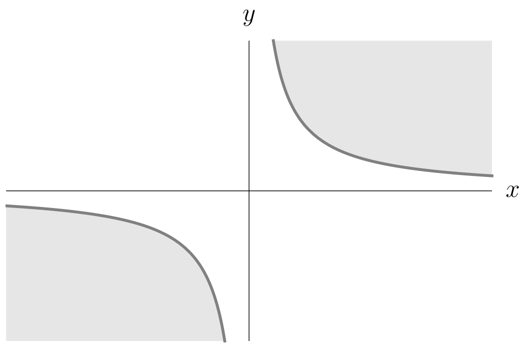


Figure 1: The set of points (shaded) $\operatorname{Im} z^2 > 2$. The boundary (thick) is not included.

**Example 3.2.** What set of points on the complex plane is defined by

$$-\frac{\pi}{2} \leq \arg(z + 1 - \mathrm{i}) \leq \frac{3\pi}{4}\,?$$

The complex number $z + 1 - \mathrm{i} = z - (-1 + \mathrm{i})$ geometrically is a vector, going from the point $-1 + \mathrm{i}$ to the point $z$. This vector must have the principal argument between $-\pi/2$ and $3\pi/4$, and hence I get the set, which is shown in Fig. 2.

Probably one of the most important sets on the plane for us is the set defined by the condition

$$|z - z_0| < r,$$

which is of course the disk or ball with the center at the point $z_0$ of radius $r$. I will often denote this set as $B_{z_0}(r) = B(z_0, r)$ (please note that the textbook uses a different notation), and call this set as a *neighborhood* of $z_0$ of radius $r$.

Now I can define an *open* set.

Math 452: *Complex Analysis* by Artem Novozhilov©
e-mail: artem.novozhilov@ndsu.edu. Spring 2019

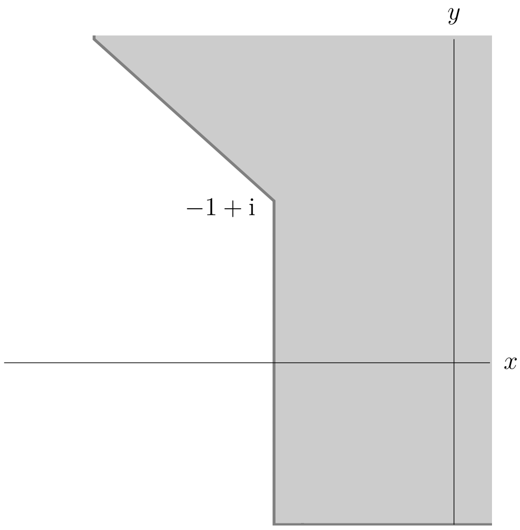


Figure 2: The set of points (shaded) in Example 3.2. The boundary (thick) is included.

**Definition 3.3.** *The set $E \subseteq \mathbf{C}$ is called open if for any point $z \in E$ there exists $r > 0$, that may depend on $z$, such that the whole neighborhood $B(z, r) \subseteq E$.*

In other words a set is open if every point in this set is internal. The examples of open sets are 1) the whole complex plane; 2) a neighborhood of radius $r$ (disk without boundary); 3) all plane without one removed point, the set in Example 3.1. The set in Example 3.2 is not open.

**Problem 3.1.** Prove carefully that any neighborhood $B(z, r)$ is an open set.

**Definition 3.4.** *A point $z_0$ is said to be a limit point of a set $E$ if every neighborhood of $z_0$ contains*

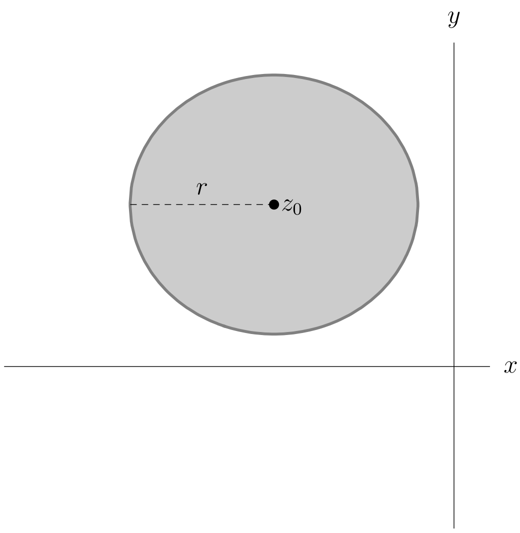


Figure 3: Neighborhood of $z_0$ or radius $r$ defined by the condition $|z - z_0| < r$. The boundary of the disk is not included in the neighborhood.

*infinitely many (distinct) points of E.*

A set $E$ is said to be *bounded* if there exists a $\rho > 0$ such that $|z| < \rho$ for all points in $E$. I state without proof an important fact that any bounded infinite set $E$ has at least one limit point. Finally, a set $F$ is said to be *closed* if it either contains all its limit points, or has no limits points at all (such as finite sets, the empty set, the set of integers, etc.)

For instance the set of all the limit points of the neighborhood $B(z, r)$ will be denoted $\overline{B}(z, r) = B(z, r) \cup \partial B(z, r)$, where $\partial B(z, r)$ is the boundary of the disk $B(z, r)$, i.e., the circle in our case. Using formulas $\overline{B}(z, r) = \{z' \colon |z' - z| \leq r\}$. Convince yourself that the set in Example 3.2 is closed.

In addition to the notion of open and closed sets we will need a very basic understanding of *connected* sets. Specifically, set $E$ is called *connected* if any of its points $z, w \in E$ can be connected by a continuous curve, all points of which are also in $E$. To be more precise, I assume an existence of the parametric curve

$$\begin{cases} x(t) = \varphi(t) \\ y(t) = \psi(t) \end{cases}, \quad t_1 \leq t \leq t_2,$$

such that $\varphi, \psi$ are continuous functions of variable $t$, $z = x(t_1) + \mathrm{i}y(t_1), w = x(t_2) + \mathrm{i}y(t_2)$, and for any $t$ satisfying $t_1 \leq t \leq t_2$, $(x(t), y(t)) \in E$.

A connected set $E \subseteq \mathbf{C}$ is called *simply connected* if for any simple closed curve (here *closed* means that the end of the curve coincides with its beginning, and *simple* means without self intersections, often such a simple closed curve is called *Jordan's curve*) that belong to this set, the set of all its *interior* points also belongs to $E$. The set is called multi connected if it is not simply connected.

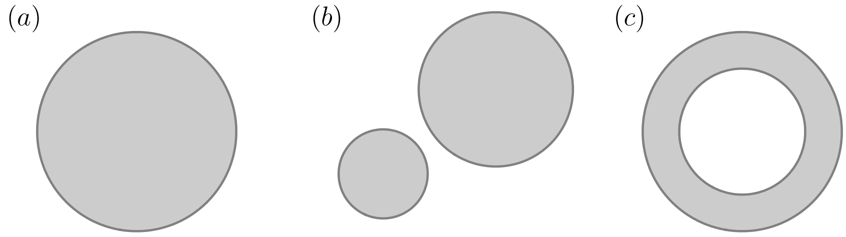


Figure 4: Examples of $(a)$ Connected and simply connected set, $(b)$ disconnected set, $(c)$ connected but not simply connected set.

Finally, I define *domain E* as an open and connected set (the textbook uses the word "region").

## 3.2 Even more geometry

Here I will consider a couple more problems that involve geometric interpretation of complex numbers.

**Example 3.5.** What curve is defined by the equation

$$|z + c| + |z - c| = 2a, \quad c \in \mathbf{R}, \quad a \in \mathbf{R}, \quad a > c?$$

I can answer this question by writing $z = x + \mathrm{i}y$, etc. but a better way is to notice that $|z + c|$ is the distance from $z$ to $-c$ and $|z - c|$ is the distance from $z$ to $c$. According to the equation the sum of these distances is constant . Therefore, this curve is an ellipse, which has the equation

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, \quad b^2 = a^2 - c^2.$$

**Example 3.6.** What curve is defined by

$$\operatorname{Re}\frac{1}{z}=\frac{1}{4}?$$

Here probably the algebraic approach is the simplest one. I have

$$\operatorname{Re}\frac{1}{z}=\frac{\frac{1}{\bar z}+\frac{1}{z}}{2}=\frac{z+\bar z}{2|z|^2}=\frac{x}{x^2+y^2}\,,$$

hence

$$x^2+y^2-4x=0,$$

which is the circle $(x-2)^2+y^2=4$.

**Example 3.7.** Write in complex form the equation of the straight line

$$Ax+By+C=0.$$

Note that we can always use $x=(\bar z+z)/2, y=\mathrm{i}(\bar z-z)/2$, hence

$$(A+\mathrm{i}B)\bar z+(A-\mathrm{i}B)z+2C=0,$$

or, using $a=A+\mathrm{i}B$,

$$a\bar z+\bar a z+2C=0.$$

**Problem 3.2.** Let $z_1, z_2, z_3$ be given three points; $t_1, t_2, t_3$ be given three positive numbers such that $t_1+t_2+t_3=1$. Show that points $\xi=t_1z_1+t_2z_2+t_3z_3$ are inside the triangle $z_1z_2z_3$.

**Example 3.8.** Three circles with the centers at $O_1, O_2, O_3$ of the same radius intersect at the same point $O$. Let $A_1, A_2, A_3$ be the other points of their intersections. Show that the triangles $O_1O_2O_3$ and $A_1A_2A_3$ are equal. Here an immediate observation, if I put the origin at the point $O$ and denote the coordinates of the centers as $z_1, z_2, z_3$, is that the second point $A_1$ of intersection of circles with centers $O_1, O_2$ can be found as $z_1+z_2$, the second point $A_2$ of intersection of circles with centers $O_1, O_3$ is $z_1+z_3$, and the second point $A_3$ of intersection of circles with centers $O_2, O_3$ is $z_2+z_3$ (make a sketch if something is not clear here). Now I have, for instance, that $|A_1A_2|=|z_1+z_2-z_1-z_3|=|z_2-z_3|=|O_2O_3|$, similarly I'll find that two other sides are equal to $O_1O_3$ and $O_1O_2$. Hence the triangles are equal.